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# On vector Hankel determinants

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**Abstract**

Recently, a definition of Hankel determinants $H_k^n$ whose entries belong to a real finite dimensional linear space $\mathbb{R}^d$ has been given. This definition is based on designants and Clifford algebra. Such determinants appear in the theory of vector orthogonal polynomials, vector Padé approximants, in the algebraic approach to the vector $\varepsilon$-algorithm and other areas. Its fundamental algebraic property is that it is a vector of the real linear space $\mathbb{R}^d$. Sylvester's identity is still valid for computing recursively these determinants, involving elements of Clifford algebra. The aim of this paper is to show that this way (Sylverster's identity) is not an optimal one and to propose a more effecient alternative one, since it avoids the use of the Clifford algebra structure. This new identity will be also called Sylvester's identity since it is equivalent to the classical Sylvester's identity in the scalar case. It allows us also to recover the fundamental property more easily. Moreover, an expression of $H_k^n$ in terms of classical determinants will be given and also some new determinantal identities. © 2000 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

Let $\mathscr{M}_m(\mathbb{K})$ be the set of square matrices $m \times m$, whose coefficients are in $\mathbb{K}$, where $\mathbb{K}$ is a commutative field (or ring). A classical determinant is a multilinear

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application (denoted here by *det*) from $\mathscr{M}_m(\mathbb{K})$ onto $\mathbb{K}$ satisfying the two properties: (i) $\det(A) = 0 \Longleftrightarrow A$ singular and (ii) $\det(AB) = \det(A) \cdot \det(B)\ \forall A,\ B \in \mathscr{M}_m(\mathbb{K})$. However, when $\mathbb{K}$ is non-commutative, classical determinants do not exist. In fact, Dyson's theorem [12,16] states that if there exists a classical determinant on $\mathscr{M}_m(\mathbb{K})$, then the multiplicative law on $\mathbb{K}$ is necessarily commutative. To remedy this, Dieudonné [11] defined an application $\mathscr{M}_m(\mathbb{K}) \longrightarrow (\mathbb{K}^*/C) \cup \{0\}$, where $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ and $C$ is the group of commutators of $\mathbb{K}^*$ obeying the conditions above. When $\mathbb{K}$ is commutative, then $(\mathbb{K}^*/C) \cup \{0\}$ is isomorphic to $\mathbb{K}$ and the application coincides with the classical determinant. In the non-commutative case, $(\mathbb{K}^*/C) \cup \{0\}$ cannot be identified with $\mathbb{K}$ and the value of the Dieudonné application is an equivalence class. This result shows the important rôle played by the commutativity of the multiplicative law in the theory of determinants. It is shown in [28] that the definition of vector Hankel determinants is closely connected with determinants whose entries belong to a non-commutative ring. Non-commutative determinants were a subject of interest for many people, among them Cayley [9], Richardson [21], Dieudonné [11], and Ore [18] and it is still of actuality [2,12,16,19,20]. In [28], designants are used to define a Hankel determinant whose entries belong to a linear space $\mathbb{R}^d$, obeying to three principles:

(i) it must belong to the same finite dimensional linear space $\mathbb{R}^d$,
(ii) if the determinant vanishes, then the matrix is non-invertible,
(iii) as for the classical determinant, a recursive rule allowing easy computations is required.

Designants were proposed by Heyting in 1927 [14] for solving linear systems in a non-commutative field (or ring). They correspond to Gaussian elimination. Using Clifford algebra and designants, a definition of determinants whose coefficients are vectors of $\mathbb{R}^d$ was given in [28]. It was also explained that this definition has no interest in the general case (that means when entries are arbitrary vectors of $\mathbb{R}^d$) since the designant can be an arbitrary element of Clifford algebra. A general element of Clifford algebra is represented by a matrix of dimension $2^d \times 2^d$. However it was shown in [28] that this definition becomes very interesting for vector Hankel determinants. The fundamental result is that the designant is also a vector of $\mathbb{R}^d$. Sylverster's identity for designants [14] was used to prove this result and considered as a recursive rule for computations. It has a crucial drawback: it involves computations in the Clifford algebra. This is a real handicap from the practical point of view. The aim of this paper is not only to give an easier proof of the fundamental result of [28], but also to derive an efficient algorithm: all computations will be made in the real linear space $\mathbb{R}^d$ itself, thus avoiding the computations in the Clifford algebra. The algorithm is an identity between a vector Hankel determinant of order $k + 1$ and four vector Hankel determinants of order $k$. This identity will be called also Sylvester's identity since, in the scalar case, it is equivalent to the classical Sylvester's identity. Moreover, an expression in terms of classical determinants will be given. This expression allows us to give new identity for classical determinants. Thus the Clifford algebra becomes only a theoretical support.

To end this section, let us give the definition of designants. More details and properties can be found in [14,23–25,28].

Consider the system of homogeneous linear equations in the $n$ unknowns $x_1, x_2, \ldots, x_n$, with coefficients on the right

$$\begin{cases} x_1 a_{11} & + & x_2 a_{12} & + & \cdots & + & x_n a_{1n} = 0, \\ x_1 a_{21} & + & x_2 a_{22} & + & \cdots & + & x_n a_{2n} = 0, \\ \vdots & \vdots & & \ddots & \vdots & & \\ x_1 a_{n1} & + & x_2 a_{n2} & + & \cdots & + & x_n a_{nn} = 0, \end{cases} \tag{1}$$

where $a_{ij} \in \mathbb{K}$ for $i, j = 1, \ldots, n$.

By eliminating $x_1$ from the $(n-1)$ last equations, then $x_2$ from the $(n-2)$ last equations, and so on, we obtain $x_n \Delta^{(n)} = 0$.

$\Delta^{(n)}$ is called the right designant of system (1) and it is denoted by

$$\Delta^{(n)} = \Delta \begin{bmatrix} 1,\ldots,n \\ 1,\ldots,n \end{bmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_r .$$

$\Delta^{(n)}$ has a meaning only if its principal minors

$$\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(n-1)},$$

are all invertible. System (1) has one and only one solution if $\Delta^{(n)}$ is invertible. Similarly, one can build the left designant $\Gamma^{(n)}$ of the system of homogeneous linear equations in the $n$ unknowns $x_1, x_2, \ldots, x_n$, with coefficients on the left

$$\begin{cases} a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n = 0, \\ a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n = 0, \\ \vdots & \vdots & & \ddots & \vdots & & \\ a_{n1} x_1 & + & a_{n2} x_2 & + & \cdots & + & a_{nn} x_n = 0, \end{cases} \tag{2}$$

where $a_{ij} \in \mathbb{K}$ for $i, j = 1, \ldots, n$.

Let us just give Sylvester's identity for designants [14] and their links with classical determinants when $\mathbb{K}$ is commutative.

Let $\Delta_{q,r}^{(p)}$ denote the right designant $\Delta \begin{bmatrix} 1,\ldots,p,r \\ 1,\ldots,p,q \end{bmatrix}$ of order $p+1$. Thus $\Delta_{p+1,p+1}^{(p)} = \Delta^{(p+1)}$. We have:

**Property 1.**

$$\Delta^{(n)} = \begin{vmatrix} \Delta_{p+1,p+1}^{(p)} & \cdots & \Delta_{p+1,n}^{(p)} \\ \vdots & \ddots & \vdots \\ \Delta_{n,p+1}^{(p)} & \cdots & \Delta_{n,n}^{(p)} \end{vmatrix}_r .$$

*Taking* $p = n-2$, *we obtain Sylvester's identity* [1,14] *for designants*

$$\Delta^{(n)} = \begin{vmatrix} \Delta^{(n-2)}_{n-1,n-1} & \Delta^{(n-2)}_{n-1,n} \\ \Delta^{(n-2)}_{n,n-1} & \Delta^{(n-2)}_{n,n} \end{vmatrix}_r .$$

Let $D^{(n)}$ be the determinant of system (1), where the field $\mathbb{K}$ is commutative, $\Delta^{(n)}$ the right designant of the same system (here $\Delta^{(n)} = \Gamma^{(n)}$). We have a relation between designants and determinant

**Property 2.** *If $\mathbb{K}$ is commutative*,

$$D^{(n)} = \Delta^{(n)} \cdot \Delta^{(n-1)} \cdots \Delta^{(2)} \cdot \Delta^{(1)}.$$

## 2. Clifford algebra and group

Let $\mathbb{R}^d$ be the real linear space of dimension $d$ and $\{e_1, \ldots, e_d\}$ the canonical orthonormal basis of $\mathbb{R}^d$. The scalar product of two vectors $x, y$ of $\mathbb{R}^d$ will be denoted by $(x|y)$. The real Clifford algebra associated to $\mathbb{R}^d$ is a unitary, associative but non-commutative (for $d > 1$) algebra $\mathscr{C}_d$ generated by $e_1, \ldots, e_d$ (see [2,10]) (thus $\mathbb{R}^d \subset \mathscr{C}_d$), which satisfies the anti-commutation relation

$$e_i e_j + e_j e_i = 2\delta_{ij}\mathbf{1}, \tag{3}$$

where $\delta_{ij}$ is Kronecker symbol and $\mathbf{1}$ is the unit element of the algebra. It is also a real linear space spanned by the products

$$e_{i_1}, \ldots, e_{i_r}, \quad 0 \leqslant i_1 < \cdots < i_r \leqslant d.$$

The system $\{\mathbf{1}, e_1, \ldots, e_d, e_1e_2, \ldots, e_1, \ldots, e_d\}$ is a basis of $\mathscr{C}_d$ and it is easy to see that its dimension is $2^d$.

There are various matrix representations of the basic elements $e_i$ (see for example [15,17,32]). A general element of $\mathscr{C}_d$ can be represented by a real matrix $2^d \times 2^d$. One can consider $\mathbb{R}^d$ and $\mathbb{R}$ as subspaces of $\mathscr{C}_d$. The anti-commutation relation (3) implies

$$\forall x \in \mathbb{R}^d \quad \forall y \in \mathbb{R}^d, \quad xy + yx = 2(x|y)\mathbf{1}. \tag{4}$$

Relation (4) has two particular cases. The first one occurs when the vector $x$ is orthogonal to the vector $y$ (that is $(x|y) = 0$). So

$$x \perp y; \quad xy = -yx. \tag{5}$$

The second one is when $x = y$,

$$xx = (x|x) = \|x\|^2\mathbf{1}. \tag{6}$$

Thus, for $x \in \mathbb{R}^d$, different from zero, its inverse is given by

$$x^{-1} = \frac{x}{\|x\|^2}. \tag{7}$$

Thus any non-zero vector $x$ of $\mathbb{R}^d$ is invertible. However, this assertion is not true for any element of $\mathscr{C}_d$. In fact, $\mathscr{C}_d$ is not a division algebra, for example,

$$1 - e_1 \neq 0, \quad 1 + e_1 \neq 0, \quad \text{and} \quad (1 - e_1)(1 + e_1) = 0.$$

From relation (4), we can deduce the following relation:

$$\forall x, y \in \mathbb{R}^d, \quad xyx = 2(x|y)x - \|x\|^2 y. \tag{8}$$

So, for all $x, y$ belonging to $\mathbb{R}^d$, the product $xyx$ belongs to $\mathbb{R}^d$.

From (4) we deduce (see [32])

$$\forall x, y, z \in \mathbb{R}^d, \quad xyz + zyx \in \mathbb{R}^d. \tag{9}$$

Relation (4) has an equivalent form which has a geometric interpretation

$$\forall x \in \mathbb{R}^d, \quad x \neq 0, \quad \forall y \in \mathbb{R}^d, \quad x^{-1}yx = \frac{2(x|y)x}{\|x\|^2} - y. \tag{10}$$

Consider a non-zero vector $a \in \mathbb{R}^d$ and the automorphism

$$\xi_a : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad y \longrightarrow a^{-1}ya = \xi_a(y) = -\tau_a(y)$$

with $\tau_a(y) = y - 2(a|y)a/\|a\|^2$. $\tau_a$ is the orthogonal symmetry with respect to the hyperplane $(\mathbb{R}a)^\perp$. In fact, $\tau_a(a) = a - 2a = -a$, and for $y \in (\mathbb{R}a)^\perp$, we have $(a|y) = 0$ and $\tau_a(y) = y$. Let $\mathscr{G}_d = \left\{\prod_{i=1}^m u_i,\ m \in \mathbb{N} \setminus \{0\},\ u_i \in \mathbb{R}^d \setminus \{0\}\right\}$. $\mathscr{G}_d$ forms a group for the multiplicative law [2].

**Definition 1.** The group $\mathscr{G}_d$ is called the Clifford group.

There exists a norm on $\mathscr{G}_d$, which is an extension of the 2-norm of $\mathbb{R}^d$, called the spinor norm, defined in the following way.

Let $\phi$ be the anti-automorphism on $\mathscr{C}_d$ defined on the basic elements by

$$\phi(\mathbf{1}) = \mathbf{1} \text{ and for } 0 < i_1 < \cdots < i_r \leqslant d, \quad \phi(e_{i_1} \cdots e_{i_r}) = e_{i_r} \cdots e_{i_1}.$$

Setting $\phi(x) = \widetilde{x}$, we immediately have

$$\forall x \in \mathscr{C}_d, \quad \forall y \in \mathscr{C}_d, \quad \widetilde{xy} = \widetilde{y}\widetilde{x}. \tag{11}$$

Let $u$ be an element of $\mathscr{G}_d$. Then there exists $u_1, \ldots, u_r \in \mathbb{R}^d \setminus \{0\}$ such that $u = u_1 \cdots u_r$. We have

$$u\widetilde{u} = u_1 \cdots u_r \widetilde{u_r} \cdots \widetilde{u_1} = \|u_1\|^2 \cdots \|u_r\|^2. \tag{12}$$

The spinor norm of $u$ is $\|u\| = \sqrt{u\widetilde{u}}$ and it follows that $u^{-1} = \widetilde{u}/\|u\|^2\ \forall u \in \mathscr{G}_d$.

## 3. Vector Hankel determinants

In the sequel, unless mentioned the contrary, $c_0, c_1, \ldots$ denotes a vector sequence ($c_i \in \mathbb{R}^d$). The main result in [28] is that the right designant

$$\begin{vmatrix} c_0 & c_1 & \dots & c_k \\ c_1 & c_2 & \dots & c_{k+1} \\ \vdots & & & \vdots \\ c_k & c_{k+1} & \dots & c_{2k} \end{vmatrix}_r$$

belongs to $\mathbb{R}^d$. The direct proof of this result in [28] takes some pages. It was based on Sylvester's identity for designants [14]. In this context, this identity presents a drawback which can be explained in the following example. For $n = 0$, we have $|c_0|_r = c_0 \in \mathbb{R}^d$. For $n = 1$, we have

$$\begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}_r = c_2 - c_1 {c_0}^{-1} c_1$$

which, from (8), belongs to $\mathbb{R}^d$. Thus, the assertion is easy to prove for $n = 0$ or $n = 1$. Let us see what happens for $n = 2$. We set

$$t = \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}_r, \quad u = \begin{vmatrix} c_0 & c_2 \\ c_2 & c_4 \end{vmatrix}_r, \quad v = \begin{vmatrix} c_0 & c_2 \\ c_1 & c_3 \end{vmatrix}_r,$$
$$w = \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}_r, \quad x = \begin{vmatrix} c_0 & c_1 \\ c_2 & c_3 \end{vmatrix}_r.$$

By Sylvester's identity (Property 1), we have $t = u - vw^{-1}x$. Since $x = \widetilde{v}$, we obtain $t = u - vw^{-1}\widetilde{v}$. Now we can see that $u$ and $w$ belong to $\mathbb{R}^d$ (these results correspond to the case $n = 1$ above). If $v \in \mathbb{R}^d$, then $\widetilde{v} = v$ and Eq. (8) allows us to say that $vw^{-1}\widetilde{v} \in \mathbb{R}^d$. The problem is that, in general, $v$ does not belong to $\mathbb{R}^d$ since $v = c_3 - c_2 {c_0}^{-1} c_1$ and $c_2 {c_0}^{-1} c_1$ is not necessarily in $\mathbb{R}^d$. Thus structure of $\mathscr{C}_d$ is needed for computing $v$. For a practical point of view, this is a real drawback since the elements of Clifford algebra are represented by quite large matrices ($2^d \times 2^d$). The aim of this paper is to give another identity which leads to a very easy proof that a Hankel determinant is a vector of $\mathbb{R}^d$ and avoids the use of the Clifford algebra for the recursive computations. All computations will be made with the classical operations on the real linear space $\mathbb{R}^d$. We set

$$H_k^n = \begin{vmatrix} c_n & \dots & c_{n+k} \\ \vdots & & \vdots \\ c_{n+k} & \dots & c_{n+2k} \end{vmatrix}_r.$$

Thus, we have, for the preceding example, $t = H_2^0$. It will be proved in the sequel that

$$H_2^0 = H_1^2 + H_1^1 \left( [H_0^2]^{-1} - [H_1^0]^{-1} \right) H_1^1.$$

Since $H_1^2,\ H_1^1,\ H_0^2,\ H_1^0 \in \mathbb{R}^d$, we obtain $H_2^0 \in \mathbb{R}^d$ and we have a procedure for computing it without using the Clifford algebra. We will now present the proof in the general case. It is a consequence of the theory of vector orthogonal polynomials.

## 4. Formal vector orthogonal polynomials

Let $\mathscr{P}$ be the set of polynomials in one real variable and whose coefficients belong to $\mathscr{C}_d$ and $\mathscr{P}_k$ the set of elements of $\mathscr{P}$ of degree $i \leqslant k$. For $n \in \mathbb{N}$, let $l^{(n)}$ be the left $\mathscr{C}_d$-linear functional defined by

$$l^{(n)} : \mathscr{P} \longrightarrow \mathscr{C}_d, \quad \lambda x^j \longrightarrow l^{(n)}(\lambda x^j) = c_{n+j}\lambda \quad \forall \lambda \in \mathscr{C}_d. \tag{13}$$

The vector sequence $(c_i)_{i=0}^{\infty}$ is called the sequence of moments.

Let $L_k^n(x) = \sum_{i=0}^{k} \alpha_i^{k,n} x^i$, where $\alpha_k^{k,n} = \mathbf{1}$ such that

$$l^{(n)}(x^j L_k^n) = 0, \quad j = 0, \ldots, k-1. \tag{14}$$

Obviously, we obtain

$$l^{(n)}(L_k^n p) = 0 \quad \forall p \in \mathscr{P}_{k-1}. \tag{15}$$

**Definition 2.** If $L_k^n$ exists, it is said to be the formal vector orthogonal polynomial of degree $k$ with respect to $l^{(n)}$.

Eq. (14) can be rewritten as

$$\left\{\begin{array}{ccccccc} c_n\alpha_0^{k,n} & + & \ldots & + & c_{k-1+n}\alpha_{k-1}^{k,n} & = & -c_{k+n} \\ \vdots & & & & \vdots & & \vdots \\ c_{j+n}\alpha_0^{k,n} & + & \ldots & + & c_{k-1+j+n}\alpha_{k-1}^{k,n} & = & -c_{k+j+n} \\ \vdots & & & & \vdots & & \vdots \\ c_{k-1+n}\alpha_0^{k,n} & + & \ldots & + & c_{2k-2+n}\alpha_{k-1}^{k,n} & = & -c_{2k-1+n} \end{array}\right. \tag{16}$$

and $L_k^n$ is given by solving this system. If the Hankel determinant $H_{k-1}^n$ exists and is invertible, then $L_k^n$ exits and is unique.

These polynomials $L_k^i$ can be displayed in a two-dimensional array $(L)$

$$\begin{array}{ccccc} L_{-1}^0 & & & & \\ L_{-1}^1 & L_0^0 & & & \\ L_{-1}^2 & L_0^1 & L_1^0 & & \\ L_{-1}^3 & L_0^2 & L_1^1 & L_2^0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

$l^{(n)}$ is said to be definite if all the orthogonal polynomials $L_k^{(n)}$ exist for $k = 1, 2, \ldots$ It is shown in [26] that $l^{(n)}$ is definite if and only if $\forall k \in \mathbb{N},\ H_{k-1}^n \neq 0$.

As in the scalar case, it is shown in [26] that the orthogonal polynomial $L_k^n$ can be expressed as a designant directly in terms of moments

$$\forall k \in \mathbb{N} \setminus \{0\} \quad \forall n \in \mathbb{N}, \quad L_k^n(x) = \left|\begin{array}{cccc} c_n & \cdots & c_{k+n-1} & 1 \\ \vdots & & \vdots & \vdots \\ c_{k+n} & \cdots & c_{2k+n-1} & x^k \end{array}\right|_r. \tag{17}$$

From this designantal formula, we deduce

$$\forall k \in \mathbb{N} \setminus \{0\} \quad \forall n \in \mathbb{N}, \quad l^{(n)}(x^k L_k^n(x)) = H_k^n. \tag{18}$$

The orthogonal polynomials $L_{k+1}^n,\ L_k^n,\ L_{k-1}^n$ satisfy a three-term recurrence relationship

$$L_{k+1}^n(x) = L_k^n(x)(x - q_{k+1}^n - e_k^n) - L_{k-1}^n(x) q_k^n e_k^n. \tag{19}$$

We also have

$$x L_k^{n+1}(x) = L_{k+1}^n(x) + L_k^n(x) q_{k+1}^n, \tag{20}$$

$$\forall k \in \mathbb{N}, \quad L_k^{n+1}(x) = L_k^n(x) - L_{k-1}^{n+1}(x) e_k^n, \tag{21}$$

with the conventions $e_0^n = 0,\ -q_0^n e_0^n = c_n$, and

$$q_{k+1}^n = [H_k^n]^{-1} H_k^{n+1}, \tag{22}$$

$$e_k^n = [H_{k-1}^{n+1}]^{-1} H_k^n. \tag{23}$$

It is shown that the *qd*-algorithm allows us to compute recursively the quantities $q_k^n$ and $e_k^n$. It is given by the following rules (see [26] for the proof)

$$\begin{cases} q_{k+1}^{n+1} + e_k^{n+1} = q_{k+1}^n + e_{k+1}^n, \\ e_k^n q_{k+1}^n = q_k^{n+1} e_k^{n+1}. \end{cases} \tag{24}$$

These quantities will be displayed in a two-dimensional array as follows:

$$\begin{array}{ccccccc}
e_0^0 & & & & & & \\
 & q_1^0 & & & & & \\
e_0^1 & & e_1^0 & & & & \\
 & q_1^1 & & q_2^0 & & & \\
e_0^2 & \vdots & e_1^1 & \vdots & e_2^0 & & \\
\vdots & & \vdots & & \vdots & \ddots &
\end{array}$$

The columns of the $(qd)$ table are computed from the first column $e_0^n$ and the second column $q_1^n = [c_n]^{-1} c_{n+1}$.

**Theorem 1.** *We have*

$$\forall k \in \mathbb{N} \setminus \{0\} \quad \forall n \in \mathbb{N}, \quad H_k^n \in \mathbb{R}^d \tag{25}$$

*and $H_k^n$ can be computed recursively in $\mathbb{R}^d$ by*

$$H_{k+1}^n = H_k^{n+2} + H_k^{n+1} \left( \left( H_{k-1}^{n+2} \right)^{-1} - \left( H_k^n \right)^{-1} \right) H_k^{n+1}. \tag{26}$$

*with the initialisations $H_{-1}^n = \infty,\ H_0^n = c_n$.*

*This equality is equivalent to the classical Sylvester's identity in the scalar case.*

**Proof.** We have, from relation (24),

$$e_{k+1}^n = q_{k+1}^{n+1} + e_k^{n+1} - q_{k+1}^n.$$

Rewriting $e_k^n$ and $q_k^n$ in terms of $H_k^n$ we obtain immediately

$$H_{k+1}^n = H_k^{n+2} + H_k^{n+1}\left(\left(H_{k-1}^{n+2}\right)^{-1} - \left(H_k^n\right)^{-1}\right)H_k^{n+1}.$$

Using Eq. (8) and the fact that $H_0^n,\ H_1^n \in \mathbb{R}^d$, we prove by induction that $H_k^n \in \mathbb{R}^d$. $\square$

The direct proof of this result in [28] is a long one, taking some pages, and using some sophisticated properties of designants. In [28], the only recursive relation used is Sylvester's identity for designants. It is really not well adapted here since it involves computations in $\mathscr{C}_d$. From a practical point of view, this relation cannot be used since, as said above, the matrices representing elements of $\mathscr{C}_d$ are quite large ($2^d \times 2^d$). The second part of the theorem gives an alternative. The vector Hankel determinant $H_{k+1}^n$ of order $k+1$ can be computed from four vector Hankel determinants of order $k$ using only the classical operations in $\mathbb{R}^d$. Notice that, in the scalar case (that is when the sequence of moments are scalar), by using Property 2 linking determinants and designants, (26) is equivalent to the classical Sylvester's identity. In fact, denoting by

$$h_k^n = \begin{vmatrix} c_n & \dots & c_{n+k} \\ \vdots & & \vdots \\ c_{n+k} & \dots & c_{n+2k} \end{vmatrix},$$

the classical Hankel determinant, from Property 2, we have $h_k^n/h_{k-1}^n = H_k^n$. Substituting in (26), we obtain

$$\frac{h_{k+1}^n}{h_k^n} = \frac{h_k^{n+2}}{h_{k-1}^{n+2}} + \frac{h_k^{n+1}}{h_{k-1}^{n+1}}\left(\frac{h_{k-2}^{n+2}}{h_{k-1}^{n+2}} - \frac{h_{k-1}^n}{h_k^n}\right)\frac{h_k^{n+1}}{h_{k-1}^{n+1}}.$$

Substracting the first term of the right-hand side and dividing by $(h_k^{n+1})^2$, we obtain

$$\left(h_k^{n+1}\right)^{-2}\left(\frac{h_{k+1}^n}{h_k^n} - \frac{h_k^{n+2}}{h_{k-1}^{n+2}}\right) = \left(h_{k-1}^{n+1}\right)^{-2}\left(\frac{h_{k-2}^{n+2}}{h_{k-1}^{n+2}} - \frac{h_{k-1}^n}{h_k^n}\right).$$

Notice that the right-hand side is exactly the left-hand side if instead of $k$ one takes $k-1$. Setting $k=2$ in the right-hand side, then it is equal to $-1$ and we deduce

$$h_{k+1}^n h_{k-1}^{n+2} = h_k^{n+2} h_k^n - \left[h_k^{n+1}\right]^2.$$

This is Sylvester's identity for Hankel determinants (see [4]).

## 5. Expression of $H_k^n$ in term of classical determinants

This expression is based on the theory of generalized inverse Padé approximants [13] and their links with formal vector orthogonal plynomials and vector Padé approximants [26,27]. There is no loss of generality by taking $n = 0$. Setting $H_k = H_k^0,\ L_k = L_k^0$ and

$$P_{2k}(x) = \frac{\begin{vmatrix} 1 & \cdots & x^{2k} \\ \nu_{00} & \cdots & \nu_{0,2k} \\ \vdots & & \vdots \\ \nu_{2k-1,0} & \cdots & \nu_{2k-1,2k} \end{vmatrix}}{\begin{vmatrix} \nu_{00} & \cdots & \nu_{0,2k-1} \\ \vdots & \ddots & \vdots \\ \nu_{2k-1,0} & \cdots & \nu_{2k-1,2k-1} \end{vmatrix}},$$

where the scalar $\nu_{ij}$ are given by

$$\nu_{ii} = 0, \quad i = 0, 1, \ldots, 2k-1, \tag{27}$$

$$\nu_{ij} = \sum_{k=0}^{j-i-1} (c_{i+k} | c_{j-k-1}), \quad j = i+1, \ldots, 2k, \tag{28}$$

$$\nu_{ij} = -\sum_{k=0}^{j-i-1} (c_{j+k} | c_{i-k-1}), \quad j = 0, \ldots, i-1. \tag{29}$$

Thus

$$\nu_{ij} = -\nu_{ji}, \quad i, j = 0, \ldots, 2k-1. \tag{30}$$

The polynomial $P_{2k}$ is monic and of degree $2k$ provided that the even skew determinant $|\nu_{ij}|_{i,j=0}^{2k-1}$ is non-zero. Notice that this determinant is positive and that its square root is a Pfaffian [1].

**Theorem 2.** *We have*

$$H_k = \frac{\begin{vmatrix} c_0 & \cdots & c_{2k} \\ \nu_{00} & \cdots & \nu_{0,2k} \\ \vdots & & \vdots \\ \nu_{2k-1,0} & \cdots & \nu_{2k-1,2k} \end{vmatrix}}{\begin{vmatrix} \nu_{00} & \cdots & \nu_{0,2k-1} \\ \vdots & \ddots & \vdots \\ \nu_{2k-1,0} & \cdots & \nu_{2k-1,2k-1} \end{vmatrix}}. \tag{31}$$

**Proof.** Using the theory of vector orthogonal polynomials and vector Padé approximants [22,26,27] and by uniqueness property, we deduce $L_k(x)\widetilde{L}_k(x) = P_{2k}(x)$.

Applying $l^{(0)}$ to the left-hand side, we obtain $l^{(0)}\left(L_k(x)\widetilde{L}_k(x)\right) = l^{(0)}\left(x^k L_k(x)\right) = H_k$ and then to the right-hand side, we obtain $H_k = l^{(0)}(P_{2k}(x))$, i.e.,

$$H_k = \frac{\begin{vmatrix} c_0 & \cdots & c_{2k} \\ v_{00} & \cdots & v_{0,2k} \\ \vdots & & \vdots \\ v_{2k-1,0} & \cdots & v_{2k-1,2k} \end{vmatrix}}{\begin{vmatrix} v_{00} & \cdots & v_{0,2k-1} \\ \vdots & \ddots & \vdots \\ v_{2k-1,0} & \cdots & v_{2k-1,2k-1} \end{vmatrix}}. \qquad \square$$

Thus, we have an expression of $H_k$ involving classical determinants. Elements of the first row are vectors of $\mathbb{R}^d$. The determinant is obtained by expanding it with respect to its first row by using the classical rule for expanding a determinant. The expression of $H_k^n$ comes out from $H_k$ by replacing $c_l$ by $c_{l+n}$ in the expression of $H_k$. The fundamental property becomes obvious. Eq. (26) gives a new identity for determinants. In particular, when the moments $c_i$ are scalar, from Property 2, we have $H_k^n = h_k^n / h_{k-1}^n$ and then substituting in (31), we obtain:

**Corollary 1.** *Let $(c_i)_{i=0}^{\infty}$ be a real sequence. We have*

$$\begin{aligned} &\left|c_{i+j}\right|_{i,j=0}^{k} \begin{vmatrix} v_{00} & \cdots & v_{0,2k-1} \\ \vdots & \ddots & \vdots \\ v_{2k-1,0} & \cdots & v_{2k-1,2k-1} \end{vmatrix} \\ &\quad = \left|c_{i+j}\right|_{i,j=0}^{k-1} \begin{vmatrix} c_0 & \cdots & c_{2k} \\ v_{00} & \cdots & v_{0,2k} \\ \vdots & & \vdots \\ v_{2k-1,0} & \cdots & v_{2k-1,2k} \end{vmatrix}. \end{aligned} \tag{32}$$

**Example 1.** For $k = 1$, we have

$$\begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix} \begin{vmatrix} 0 & c_0^2 \\ c_0^2 & 0 \end{vmatrix} = c_0 \begin{vmatrix} c_0 & c_1 & c_2 \\ 0 & c_0^2 & 2c_0c_1 \\ -c_0^2 & 0 & c_1^2 \end{vmatrix}.$$

## 6. Conclusion

It is well known that scalar Hankel determinants play an important rôle in the theory of scalar orthogonal polynomials (see for example [3,4]). It is shown in [8] that their use provides an interesting framework for Lanczos methods for solving linear systems. In particular, the methods such as Lanczos/Orthomin, Lanczos/Orthodir and Lanczos/Orthores [30,31] correspond to some recurrence relations for orthogonal polynomials and are equivalent to the topological $\varepsilon$-algorithm [8]. A determinantal expression for the conjugate gradient squared (CGS) of Sonneveld [29] was given. Moreover, treatments of breakdowns and near-breakdowns in the CGS and Lanczos-type product methods were given in [5–7]. In a forthcoming paper, we will show that the theory of vector orthogonal polynomials and vector Hankel determinant is helpful and allow to derive new results. In particular, a new determinantal formula for the CGS and a determinantal formula for the vector $\varepsilon$-algorithm (VEA) when applied to linear systems will be given. These formulae allow to compare the two algorithms. We show that the vector Hankel determinant is nothing but the residual of the VEA when applied to linear systems. Problems of breakdowns and near-breakdowns become more accessible and are under study.

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